

Subgroups of polynomial automorphisms with diagonalizable fibers

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Abstract

Let R be an integral domain over a field k , and G a subgroup of the automorphism group of the polynomial ring $R[x_1, \dots, x_n]$ over R . In this paper, we discuss when G is diagonalizable under the assumption that G is diagonalizable over the field of fractions of R . We are particularly interested in the case where G is a finite abelian group. Kraft-Russell (2014) implies that every finite abelian subgroup of $\text{Aut}_R R[x_1, x_2]$ is diagonalizable if R is an affine PID over $k = \mathbf{C}$. One of the main results of this paper says that the same holds for a PID R over any field k containing enough roots of unity.

1 Introduction

For each commutative ring R , we denote by $R[\mathbf{x}] = R[x_1, \dots, x_n]$ the polynomial ring in n variables over R , and by $\text{Aut}_R R[\mathbf{x}]$ the automorphism group of the R -algebra $R[\mathbf{x}]$. We identify an endomorphism ϕ of the R -algebra $R[\mathbf{x}]$ with the n -tuple $(\phi(x_1), \dots, \phi(x_n))$ of elements of $R[\mathbf{x}]$, where the composition is defined by $\phi \circ \psi = (\phi(\psi(x_1)), \dots, \phi(\psi(x_n)))$. Note that, if G is a subgroup of $\text{Aut}_R R[\mathbf{x}]$, and S is a commutative R -algebra, then $G_S := \{\text{id}_S \otimes \phi \mid \phi \in G\}$ is a subgroup of $\text{Aut}_S S[\mathbf{x}]$. When $S = \kappa(\mathfrak{p})$ is the residue field of the localization $R_{\mathfrak{p}}$ of R at a prime ideal \mathfrak{p} of R , we denote this group by $G_{\mathfrak{p}}$. If R is a domain, K denotes the field of fractions of R .

Throughout this paper, let k be an arbitrary field. If R is a k -algebra, then $D_n(k) := \{\delta_{\mathbf{a}} \mid \mathbf{a} \in (k^*)^n\}$ is a subgroup of $\text{Aut}_R R[\mathbf{x}]$, where we define $\delta_{\mathbf{a}} := (a_1 x_1, \dots, a_n x_n)$ for each $\mathbf{a} = (a_1, \dots, a_n) \in (k^*)^n$. We say that a subgroup G of $\text{Aut}_R R[\mathbf{x}]$ is *diagonalizable* if there exists $\psi \in \text{Aut}_R R[\mathbf{x}]$ such that $\psi^{-1} \circ G \circ \psi$ is contained in $D_n(k)$.

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Now, assume that R is a k -domain. In this paper, we discuss the following problems.

Problem 1. *Let G be a subgroup of $\text{Aut}_R R[\mathbf{x}]$ such that $G_{(0)}$ is diagonalizable. Does it follow that G is diagonalizable?*

If we regard $\text{Aut}_R R[\mathbf{x}]$ as a subgroup of $\text{Aut}_K K[\mathbf{x}]$, then the assumption of Problem 1 is equivalent to $\psi^{-1} \circ G \circ \psi \subset D_n(k)$ for some $\psi \in \text{Aut}_K K[\mathbf{x}]$. When $n = 2$, this condition implies that $G_{\mathfrak{p}}$ is diagonalizable for any prime ideal \mathfrak{p} of R by van der Kulk [7] and Serre [14] (cf. Section 2). So we also consider the following problem for $n \geq 3$.

Problem 2. *Let G be a subgroup of $\text{Aut}_R R[\mathbf{x}]$ such that $G_{\mathfrak{p}}$ is diagonalizable for all the prime ideals \mathfrak{p} of R . Does it follow that G is diagonalizable?*

We are particularly interested in the case where G is a finite abelian group. In fact, whether every finite abelian subgroup of $\text{Aut}_{\mathbf{C}} \mathbf{C}[\mathbf{x}]$ is conjugate to a subgroup of $D_n(\mathbf{C})$ is a difficult problem with little progress for $n \geq 3$ (see [5] for the case $n = 2$). This problem is a special case of Kambayashi's Linearization Problem [6], and is open even for finite cyclic groups (cf. [9]). In the case of finite cyclic groups, the problem is also included in the list of "eight challenging open problems in affine spaces" by Kraft [10]. We mention that, over a field of positive characteristic, a counterexample to a similar problem is already given by Asanuma [1]. The situation is worse in the case of positive characteristic.

Under the assumptions in Problems 1 and 2, there exists a subgroup \mathcal{G} of $(k^*)^n$ for which $G_{(0)}$ is conjugate to $\{\delta_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{G}\}$ in $\text{Aut}_K K[\mathbf{x}]$. We write $\mathbf{a}^i := a_1^{i_1} \cdots a_n^{i_n}$ for each $\mathbf{a} = (a_1, \dots, a_n) \in \mathcal{G}$ and $i = (i_1, \dots, i_n) \in \mathbf{Z}^n$, and define $M_{\mathcal{G}}$ to be the set of $i \in \mathbf{Z}^n$ such that $\mathbf{a}^i = 1$ for all $\mathbf{a} \in \mathcal{G}$. Let $\gamma_1, \dots, \gamma_n$ be the images of the coordinate unit vectors of \mathbf{Z}^n in $\Gamma_{\mathcal{G}} := \mathbf{Z}^n / M_{\mathcal{G}}$. For each i , let $\Gamma_{\mathcal{G}}^{(i)}$ be the subgroup of $\Gamma_{\mathcal{G}}$ generated by γ_j for $1 \leq j \leq n$ with $j \neq i$.

The following theorem is the main result of this paper.

Theorem 1.1. (i) *When $n = 2$, Problem 1 has an affirmative answer in the following two cases:*

- (1) *R is a PID.*
- (2) *R is a regular UFD, and $\Gamma_{\mathcal{G}}^{(1)}$ or $\Gamma_{\mathcal{G}}^{(2)}$ is not equal to $\Gamma_{\mathcal{G}}$.*
- (ii) *When $n \geq 3$, Problem 2 has an affirmative answer if R is a regular UFD, and at least $n - 1$ of $\Gamma_{\mathcal{G}}^{(1)}, \dots, \Gamma_{\mathcal{G}}^{(n)}$ are not equal to $\Gamma_{\mathcal{G}}$.*

We emphasize that the base field k is arbitrary in Theorem 1.1. When R is an affine PID over $k = \mathbf{C}$, the case (1) of Theorem 1.1 (i) (and hence Corollary 1.2 to follow) is contained in Kraft-Russell [8, Thm. 3.2].

In Section 2, we derive the following corollary from the case (1) of Theorem 1.1 (i) (see the discussion after Theorem 2.3).

Corollary 1.2. *Let R be a PID over a field k , and G a finite abelian subgroup of $\text{Aut}_R R[x_1, x_2]$ with $d := \max\{\text{ord } \phi \mid \phi \in G\}$. If k contains a primitive d -th root of unity, then G is diagonalizable.*

Assume that $n = 2$. We call $f \in K[\mathbf{x}]$ a *coordinate* of $K[\mathbf{x}]$ if there exists $g \in K[\mathbf{x}]$ such that $K[f, g] = K[\mathbf{x}]$. If this is the case, for each $\phi \in \text{Aut}_K K[\mathbf{x}]$ with $\phi(f) = f$, there exists $h \in K[f]$ such that $\phi(g) = (\det J\phi)g + h$, where $J\phi$ denotes the Jacobian matrix of ϕ .

We have the following corollary to the case (2) of Theorem 1.1 (i).

Corollary 1.3. *Let R be a regular UFD over a field k , and $\phi \in \text{Aut}_R R[x_1, x_2]$ such that $\det J\phi$ belongs to $k \setminus \{1\}$. If there exists a coordinate f of $K[x_1, x_2]$ with $\phi(f) = f$, then $\langle \phi \rangle$ is diagonalizable.*

Here, $\langle \phi \rangle$ denotes the subgroup of $\text{Aut}_R R[\mathbf{x}]$ generated by ϕ . In fact, setting $u := \det J\phi$ and $\psi := (f, g + (u - 1)^{-1}h) \in \text{Aut}_K K[\mathbf{x}]$, we have $\psi^{-1} \circ \phi \circ \psi = (x_1, ux_2)$. Hence, $\psi^{-1} \circ \langle \phi \rangle \circ \psi = \{\delta_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{G}\}$ holds for $\mathcal{G} := \{(1, u^i) \mid i \in \mathbf{Z}\}$. Thus, we get $\gamma_1 = 0$, and therefore $\Gamma_{\mathcal{G}}^{(2)} \neq \Gamma_{\mathcal{G}}$.

The structure of this paper is as follows. In Section 2, we recall the notion of algebraic actions of subgroups of $(k^*)^r$ on $R[\mathbf{x}]$, and prove some preliminary results. We also derive a consequence of Theorem 1.1. In Section 3, we study centrizer of subgroups of $D_n(k)$ in $\text{Aut}_R R[\mathbf{x}]$. Section 4 is devoted to proving the case (1) of Theorem 1.1 (i). In this proof, the main result of Section 3 is crucial when k is not algebraically closed. The rest of Theorem 1.1 is proved in Section 5 using a different technique.

2 Algebraic \mathcal{G} -actions

Let \mathcal{G} be a subgroup of $(k^*)^r$, where $r \geq 1$. As in Section 1, we define $M_{\mathcal{G}}$ to be the set of $i \in \mathbf{Z}^r$ such that $\mathbf{a}^i = 1$ for all $\mathbf{a} \in \mathcal{G}$. Then, for each $\mathbf{a} \in \mathcal{G}$, the map $\mathbf{Z}^r \ni i \mapsto \mathbf{a}^i \in k^*$ factors through $\Gamma_{\mathcal{G}} := \mathbf{Z}^r / M_{\mathcal{G}}$. We denote by \mathbf{a}^{γ} the image of $\gamma \in \Gamma_{\mathcal{G}}$ under the induced map $\Gamma_{\mathcal{G}} \rightarrow k^*$. Let R be a k -algebra. An *algebraic \mathcal{G} -action* on $R[\mathbf{x}]$ is by definition a collection $V = (V_{\gamma})_{\gamma \in \Gamma_{\mathcal{G}}}$ of R -submodules of $R[\mathbf{x}]$ such that $R[\mathbf{x}] = \bigoplus_{\gamma \in \Gamma_{\mathcal{G}}} V_{\gamma}$, and $V_{\lambda} V_{\mu} \subset V_{\lambda + \mu}$ for each $\lambda, \mu \in \Gamma_{\mathcal{G}}$. For each $\mathbf{a} \in \mathcal{G}$, we define an automorphism $\phi_{\mathbf{a}}^V : R[\mathbf{x}] \rightarrow R[\mathbf{x}]$ by $\phi_{\mathbf{a}}^V(f) = \mathbf{a}^{\gamma} f$ for $f \in V_{\gamma}$ and $\gamma \in \Gamma_{\mathcal{G}}$. Then, the map

$$\rho^V : \mathcal{G} \ni \mathbf{a} \mapsto \phi_{\mathbf{a}}^V \in \text{Aut}_R R[\mathbf{x}]$$

is a homomorphism of groups. We say that $f \in R[\mathbf{x}]$ is V -homogeneous if f belongs to V_γ for some $\gamma \in \Gamma_{\mathcal{G}}$. Note that, for each $\gamma \in \Gamma_{\mathcal{G}} \setminus \{0\}$, there exists $\mathbf{a} \in \mathcal{G}$ such that $\mathbf{a}^\gamma \neq 1$. Hence, $f \in R[\mathbf{x}]$ is V -homogeneous if and only if $\phi(f) \in kf$ for all $\phi \in \rho^V(\mathcal{G})$. We say that V is *diagonalizable* if the subgroup $\rho^V(\mathcal{G})$ of $\text{Aut}_R R[\mathbf{x}]$ is diagonalizable, or equivalently there exists $\psi \in \text{Aut}_R R[\mathbf{x}]$ such that $\psi(x_1), \dots, \psi(x_n)$ are V -homogeneous. We remark that $V_\gamma \cap R = \{0\}$ holds for any $\gamma \neq 0$. If S is an R -algebra, then $V_S := (S \otimes_R V_\gamma)_{\gamma \in \Gamma_{\mathcal{G}}}$ is an algebraic \mathcal{G} -action on $S[\mathbf{x}]$ with $\rho^{V_S}(\mathcal{G}) = \rho^V(\mathcal{G})_S$.

The following lemma holds for any algebraic \mathcal{G} -action on $R[\mathbf{x}]$.

Lemma 2.1. *Let $f = \sum_{\gamma \in \Gamma_{\mathcal{G}}} f_\gamma \in R[\mathbf{x}]$, where $f_\gamma \in V_\gamma$. Then, for each $\gamma \in \Gamma_{\mathcal{G}}$, we may write f_γ as a k -linear combination of $\phi(f)$ for $\phi \in \rho^V(\mathcal{G})$.*

Proof. We prove the lemma by induction on $l := \#\{\gamma \mid f_\gamma \neq 0\}$. The assertion is clear if $l \leq 1$. Assume that $l \geq 2$. Take $\lambda, \mu \in \{\gamma \mid f_\gamma \neq 0\}$ with $\lambda \neq \mu$, and $\mathbf{a} \in \mathcal{G}$ with $\mathbf{a}^\lambda \neq \mathbf{a}^\mu$. For each $\alpha \in k^*$, we define g_α to be the sum of f_γ for $\gamma \in \Gamma_{\mathcal{G}}$ with $\mathbf{a}^\gamma = \alpha$. Then, we have $f = \sum_{\alpha \in k^*} g_\alpha$ and $\phi(g_\alpha) = \alpha g_\alpha$ for each $\alpha \in k^*$, where $\phi := \phi_{\mathbf{a}}^V$. Let $\alpha_1, \dots, \alpha_s$ be distinct elements of k^* such that $f = \sum_{i=1}^s g_{\alpha_i}$. Then, by linear algebra, each g_{α_i} is written as a k -linear combination of $\phi^j(f) = \sum_{i=1}^s \alpha_i^j g_{\alpha_i}$ for $0 \leq j < s$. Since $g_\alpha \neq 0$ for $\alpha = \mathbf{a}^\lambda, \mathbf{a}^\mu$, the number of nonzero V -homogeneous components of g_α is less than l for each α . Hence, the lemma follows by induction assumption. \square

Now, assume that $n = 2$. Recall the following fact which is a consequence of van der Kulk [7] and Serre [14] (see also [15, Prop. 1.11]): Let G be a subgroup of $\text{Aut}_k k[\mathbf{x}]$ such that $\deg G := \{\deg \phi(x_i) \mid \phi \in G, i = 1, 2\}$ is bounded above. Here, $\deg f$ denotes the total degree of f for a polynomial f . Then, G is conjugate to a subgroup of the *affine subgroup*

$$\mathfrak{A}_2(k) := \{\phi \in \text{Aut}_k k[\mathbf{x}] \mid \deg \phi(x_1) = \deg \phi(x_2) = 1\}$$

or the *Jonquière subgroup*

$$\mathfrak{J}_2(k) := \{(ax_1 + c, bx_2 + f(x_1)) \mid a, b \in k^*, c \in k, f(x_1) \in k[x_1]\}.$$

The following proposition is a consequence of this fact.

Proposition 2.2. *If k' is an extension field of k , then every algebraic \mathcal{G} -action on $k'[x_1, x_2]$ is diagonalizable.*

Proof. For $i = 1, 2$, we write $x_i = \sum_{\gamma \in \Gamma_{\mathcal{G}}} x_{i,\gamma}$, where $x_{i,\gamma} \in V_\gamma$. Then, we have $\phi_{\mathbf{a}}^V(x_i) = \sum_{\gamma \in \Gamma_{\mathcal{G}}} \mathbf{a}^\gamma x_{i,\gamma}$ for each $\mathbf{a} \in \mathcal{G}$. Hence, $\deg \rho^V(\mathcal{G})$ is bounded above by $\max\{\deg x_{i,\gamma} \mid \gamma \in \Gamma_{\mathcal{G}}, i = 1, 2\}$. By the fact above, there exists $\psi =$

$(f_1, f_2) \in \text{Aut}'_k k'[\mathbf{x}]$ such that $G' := \psi^{-1} \circ \rho^V(\mathcal{G}) \circ \psi$ is contained in $\mathfrak{A}_2(k')$ or $\mathfrak{J}_2(k')$. Write $f_i = \sum_{\gamma \in \Gamma_{\mathcal{G}}} f_{i,\gamma}$ for $i = 1, 2$, where $f_{i,\gamma} \in V_{\gamma}$. When $G' \subset \mathfrak{A}_2(k')$, we have $\phi(f_i) \in k'f_1 + k'f_2 + k'$ for each $\phi \in \rho^V(\mathcal{G})$ and $i = 1, 2$. Hence, $f_{i,\gamma}$ belongs to $k'f_1 + k'f_2 + k'$ for each i and γ by Lemma 2.1. This implies that $k'[f_{1,\lambda}, f_{2,\mu}] = k'[f_1, f_2]$ for some $\lambda, \mu \in \Gamma_{\mathcal{G}}$. Since $(f_{1,\lambda}, f_{2,\mu}) \in \text{Aut}'_k k'[\mathbf{x}]$, and $f_{1,\lambda}$ and $f_{2,\mu}$ are V -homogeneous, we conclude that V is diagonalizable. When $G' \subset \mathfrak{J}_2(k')$, we have $f_{1,\gamma} \in k'f_1 + k'$ and $f_{2,\gamma} \in k'f_2 + k'[f_1]$ for each $\gamma \in \Gamma_{\mathcal{G}}$ by Lemma 2.1. From this, the assertion follows similarly. \square

The following theorem is a consequence of the case (1) of Theorem 1.1 (i), since $\rho^V(\mathcal{G})_{(0)} = \rho^{V\kappa}(\mathcal{G})$ is diagonalizable by Proposition 2.2.

Theorem 2.3. *Let R be a PID over a field k , and \mathcal{G} a subgroup of $(k^*)^r$ for some $r \geq 1$. Then, every algebraic \mathcal{G} -action on $R[x_1, x_2]$ is diagonalizable.*

When R is an affine PID over $k = \mathbf{C}$, Theorem 2.3 is contained in Kraft-Russell [8, Thm. 3.2]. In fact, they treated actions of reductive groups more generally.

Corollary 1.2 is derived from Theorem 2.3 as follows. Let $\phi_1, \dots, \phi_r \in G$ be such that $G = \langle \phi_1 \rangle \times \dots \times \langle \phi_r \rangle$. Then, since $d_i := \text{ord } \phi_i$ is a divisor of d , there exists a primitive d_i -th root $\zeta_i \in k$ of unity for $i = 1, \dots, r$. Set $\mathcal{G} = \{(\zeta_1^{i_1}, \dots, \zeta_r^{i_r}) \mid i_1, \dots, i_r \in \mathbf{Z}\}$. Then, $\Gamma_{\mathcal{G}}$ is equal to $\prod_{i=1}^r (\mathbf{Z}/d_i\mathbf{Z})$. For each $\gamma = (\bar{i}_1, \dots, \bar{i}_r) \in \Gamma_{\mathcal{G}}$, we define V_{γ} to be the set of $f \in R[\mathbf{x}]$ for which $\phi_l(f) = \zeta_l^{i_l} f$ holds for $l = 1, \dots, r$. Then, for each $f \in R[\mathbf{x}]$, we have

$$f_{\gamma} := |G|^{-1} \sum_{l_1=0}^{d_1-1} \dots \sum_{l_r=0}^{d_r-1} \zeta_1^{-i_1 l_1} \dots \zeta_r^{-i_r l_r} (\phi_1^{l_1} \circ \dots \circ \phi_r^{l_r})(f) \in V_{\gamma}.$$

Since $f = \sum_{\gamma \in \Gamma_{\mathcal{G}}} f_{\gamma}$, we see that $V = (V_{\gamma})_{\gamma \in \Gamma_{\mathcal{G}}}$ is an algebraic \mathcal{G} -action on $R[\mathbf{x}]$ with $\rho^V(\mathcal{G}) = G$. Hence, G is diagonalizable by Theorem 2.3.

Next, let \mathcal{G} be a subgroup of $(k^*)^n$. For each $\gamma \in \Gamma_{\mathcal{G}}$, we define $R[\mathbf{x}]_{\gamma}$ to be the R -submodule of $R[\mathbf{x}]$ generated by the monomials $x_1^{i_1} \dots x_n^{i_n}$ such that the image of (i_1, \dots, i_n) in $\Gamma_{\mathcal{G}}$ is equal to γ . Then, we have

$$R[\mathbf{x}]_{\gamma} = \{f \in R[\mathbf{x}] \mid \delta_{\mathbf{a}}(f) = \mathbf{a}^{\gamma} f \text{ for all } \mathbf{a} \in \mathcal{G}\}. \quad (2.1)$$

Hence, $V = (R[\mathbf{x}]_{\gamma})_{\gamma \in \Gamma_{\mathcal{G}}}$ is an algebraic \mathcal{G} -action on $R[\mathbf{x}]$ such that $\phi_{\mathbf{a}}^V = \delta_{\mathbf{a}}$ for each $\mathbf{a} \in \mathcal{G}$.

Now, assume that R is a k -domain, and let G be a subgroup of $\text{Aut}_R R[\mathbf{x}]$ such that $G_{(0)}$ is diagonalizable. Then, there exists $\psi \in \text{Aut}_K K[\mathbf{x}]$ and a subgroup \mathcal{G} of $(k^*)^n$ such that $\psi^{-1} \circ G \circ \psi = \{\delta_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{G}\}$. When this is

the case, $V_K := (\psi(K[\mathbf{x}]_\gamma))_{\gamma \in \Gamma_{\mathcal{G}}}$ is an algebraic \mathcal{G} -action on $K[\mathbf{x}]$ satisfying $\rho^{V_K}(\mathcal{G}) = G_{(0)}$. Set

$$V_\gamma := \psi(K[\mathbf{x}]_\gamma) \cap R[\mathbf{x}] \quad \text{for each } \gamma \in \Gamma_{\mathcal{G}}. \quad (2.2)$$

We claim that $V := (V_\gamma)_{\gamma \in \Gamma_{\mathcal{G}}}$ is an algebraic \mathcal{G} -action on $R[\mathbf{x}]$ with $\rho^V(\mathcal{G}) = G$. In fact, take any $f \in R[\mathbf{x}]$ and write $f = \sum_{\gamma} f_\gamma$, where $f_\gamma \in \psi(K[\mathbf{x}]_\gamma)$. Then, by Lemma 2.1, each f_γ is a k -linear combination of $\phi(f) \in R[\mathbf{x}]$ for $\phi \in \rho^{V_K}(\mathcal{G}) = G_{(0)}$. Thus, f_γ belongs to $R[\mathbf{x}]$, and hence to V_γ . We remark that $f \in R[\mathbf{x}]$ is V -homogeneous if and only if $\phi(f) \in kf$ holds for each $\phi \in G$. In this sense, V is uniquely defined from G . If $n = 2$, then $V_{\kappa(\mathfrak{p})}$ is diagonalizable for any prime ideal \mathfrak{p} of R by Proposition 2.2. Hence, $G_{\mathfrak{p}}$ is diagonalizable as remarked after Problem 1.

Finally, we prove a lemma used in Section 5. We call a sequence f_1, \dots, f_r of elements of $R[\mathbf{x}]$ a *partial system of coordinates* of $R[\mathbf{x}]$ if there exist $f_{r+1}, \dots, f_n \in R[\mathbf{x}]$ such that $R[f_1, \dots, f_n] = R[\mathbf{x}]$, or equivalently there exists $\phi \in \text{Aut}_R R[\mathbf{x}]$ for which $\phi(x_i) = f_i$ holds for $i = 1, \dots, r$ (cf. [4, Prop. 1.1.6]). When $r = 1$, such an f_1 is called a *coordinate* of $R[\mathbf{x}]$.

Lemma 2.4. *Let R be a k -domain, and G a subgroup of $\text{Aut}_R R[\mathbf{x}]$. Assume that there exists $\psi = (f_1, \dots, f_n) \in \text{Aut}_K K[\mathbf{x}]$ with $\psi^{-1} \circ G \circ \psi \subset D_n(k)$ for which f_1, \dots, f_{n-1} form a partial system of coordinates of $R[\mathbf{x}]$. Then, G is diagonalizable.*

Proof. In the situation of the lemma, we may define a subgroup \mathcal{G} of $(k^*)^n$ and an algebraic \mathcal{G} -action V on $R[\mathbf{x}]$ as above. Then, f_1, \dots, f_{n-1} are V -homogeneous. Set $A = R[f_1, \dots, f_{n-1}]$ and $B = K[f_1, \dots, f_{n-1}]$. Then, we have $B \cap R[\mathbf{x}] = A$, since f_1, \dots, f_{n-1} is a partial system of coordinates of $R[\mathbf{x}]$. There exists $g \in R[\mathbf{x}]$ such that $R[f_1, \dots, f_{n-1}, g] = R[\mathbf{x}]$. It suffices to show that g is chosen to be V -homogeneous. Write $g = \sum_{\gamma \in \Gamma_{\mathcal{G}}} g_\gamma$, where $g_\gamma \in V_\gamma$ for each $\gamma \in \Gamma_{\mathcal{G}}$. Since g_γ belongs to $R[\mathbf{x}]$, we have $g_\gamma \in A = B \cap R[\mathbf{x}]$ if and only if $g_\gamma \in B$. We show that this holds for each $\gamma \neq \mu$, where $\mu \in \Gamma_{\mathcal{G}}$ is such that $f_n \in \psi(K[\mathbf{x}]_\mu)$. Then, it follows that $R[f_1, \dots, f_{n-1}, g_\mu] = R[\mathbf{x}]$, and the proof is complete. Observe that $B[g] = K[\mathbf{x}] = B[f_n]$. This implies that $g = uf_n + h$ for some $u \in K^*$ and $h \in B$. Write $h = \sum_{\gamma \in \Gamma_{\mathcal{G}}} h_\gamma$, where $h_\gamma \in \psi(K[\mathbf{x}]_\gamma)$. Then, we have $g_\mu = uf_n + h_\mu$, and $g_\gamma = h_\gamma$ for each $\gamma \neq \mu$. Since f_1, \dots, f_{n-1} are V -homogeneous, h_γ belongs to B for each $\gamma \in \Gamma_{\mathcal{G}}$. Therefore, g_γ belongs to B if $\gamma \neq \mu$. \square

3 Centrizer

Throughout this section, we assume that \mathcal{G} is a subgroup of $(k^*)^n$ not equal to $\{e\}$. For each $\gamma \in \Gamma_{\mathcal{G}}$, we define $R[\mathbf{x}]_\gamma$ as in Section 2, where R may

be any commutative ring for the moment. We say that $f \in R[\mathbf{x}]$ is \mathcal{G} -homogeneous if f belongs to $R[\mathbf{x}]_\gamma$ for some $\gamma \in \Gamma_{\mathcal{G}}$. Let $\phi = (f_1, \dots, f_n)$ be an element of $\text{Aut}_R R[\mathbf{x}]$. We say that ϕ is \mathcal{G} -homogeneous if f_i belongs to $R[\mathbf{x}]_{\gamma_i}$ for $i = 1, \dots, n$. We remark that, if R is a domain, and f_1, \dots, f_n are \mathcal{G} -homogeneous, then $(f_{\sigma(1)}, \dots, f_{\sigma(n)})$ is \mathcal{G} -homogeneous for some permutation $\sigma \in S_n$. Actually, since $\det J\phi$ belongs to $R[\mathbf{x}]^* = R^*$, the linear parts of f_1, \dots, f_n are linearly independent over R . Hence, there exists $\sigma \in S_n$ such that the linear monomials x_1, \dots, x_n appear in $f_{\sigma(1)}, \dots, f_{\sigma(n)}$, respectively.

Assume that R is a k -algebra. Then, in view of (2.1), we see that ϕ is \mathcal{G} -homogeneous if and only if $\delta_{\mathbf{a}}(f_i) = a_i f_i$ for all $\mathbf{a} = (a_1, \dots, a_n) \in \mathcal{G}$ and $i = 1, \dots, n$, and hence if and only if $\delta_{\mathbf{a}} \circ \phi = \phi \circ \delta_{\mathbf{a}}$ for all $\mathbf{a} \in \mathcal{G}$. Thus, the set $C_{\mathcal{G}}(R)$ of \mathcal{G} -homogeneous elements of $\text{Aut}_R R[\mathbf{x}]$ is equal to the centralizer of $\{\delta_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{G}\}$ in $\text{Aut}_R R[\mathbf{x}]$. We note that $C_{\mathcal{G}}(R)$ is a subgroup of $\text{Aut}_R R[\mathbf{x}]$ even if R does not contain k .

Now, assume that $n = 2$, and let h be a coordinate of $R[\mathbf{x}]$. Then, there exist $\phi \in \text{Aut}_R R[\mathbf{x}]$ and $i \in \{1, 2\}$ such that $\phi(x_i) = h$. If furthermore h is \mathcal{G} -homogeneous, then we have the following lemma.

Lemma 3.1. *Assume that R is a k -domain such that $R^* \cup \{0\}$ is a field. If h is a \mathcal{G} -homogeneous coordinate of $R[\mathbf{x}]$, then there exist $\phi \in C_{\mathcal{G}}(R)$ and $i \in \{1, 2\}$ such that $\phi(x_i) = h$.*

Proof. There exists $g \in R[\mathbf{x}]$ such that $R[h, g] = R[\mathbf{x}]$. We show that g is chosen to be \mathcal{G} -homogeneous. Then, (g, h) or (h, g) belongs to $C_{\mathcal{G}}(R)$ as remarked. Write $g = \sum_{\gamma \in \Gamma_{\mathcal{G}}} g_{\gamma}$, where $g_{\gamma} \in R[\mathbf{x}]_{\gamma}$ for each γ . Then, we have

$$\sum_{\gamma \in \Gamma_{\mathcal{G}}} \det J(h, g_{\gamma}) = \det J(h, g) \in R^*. \quad (3.1)$$

Put $\gamma_0 := \gamma_1 + \gamma_2 - \mu$, where $\mu \in \Gamma_{\mathcal{G}}$ is such that h belongs to $R[\mathbf{x}]_{\mu}$. Then, $\det J(h, g_{\gamma})$ belongs to $R[\mathbf{x}]_{\gamma - \gamma_0}$ for each γ . From (3.1), it follows that $\det J(h, g_{\gamma_0}) \neq 0$. This implies that h and g_{γ_0} are algebraically independent over R (cf. [4, Prop. 1.1.31]). We show that g_{γ} belongs to $R[h]$ for each $\gamma \neq \gamma_0$. Then, it follows that $R[h, g_{\gamma_0}] = R[h, g] = R[\mathbf{x}]$, and the proof is complete. Fix any $\lambda \neq \gamma_0$, and take $\mathbf{a} \in \mathcal{G}$ such that $\mathbf{a}^{\lambda} \neq \mathbf{a}^{\gamma_0}$. Then, we have

$$R[h, g] = \delta_{\mathbf{a}}(R[h, g]) = R[\delta_{\mathbf{a}}(h), \delta_{\mathbf{a}}(g)] = R[\mathbf{a}^{\mu} h, \delta_{\mathbf{a}}(g)] = R[h, \delta_{\mathbf{a}}(g)].$$

Hence, we may write $\delta_{\mathbf{a}}(g) = ug + g'$, where $u \in R^*$ and $g' \in R[h]$. Write $g' = \sum_{\gamma \in \Gamma_{\mathcal{G}}} g'_{\gamma}$, where $g'_{\gamma} \in R[\mathbf{x}]_{\gamma}$. Then, g'_{γ} belongs to $R[h]$ for each γ , since h is \mathcal{G} -homogeneous by assumption. On the other hand, from the equality

$$\sum_{\gamma \in \Gamma_{\mathcal{G}}} \mathbf{a}^{\gamma} g_{\gamma} = \delta_{\mathbf{a}}(g) = ug + g' = \sum_{\gamma \in \Gamma_{\mathcal{G}}} (ug_{\gamma} + g'_{\gamma}),$$

we see that $(\mathbf{a}^\gamma - u)g_\gamma = g'_\gamma$ holds for each γ . Since g_{γ_0} and h are algebraically independent over R , this implies that $\mathbf{a}^{\gamma_0} = u$. By the choice of \mathbf{a} , it follows that $\mathbf{a}^\lambda \neq u$. Since \mathbf{a}^λ and u are units of R , and $R^* \cup \{0\}$ is a field by assumption, we know that $\mathbf{a}^\lambda - u$ is a unit of R . Therefore, $g_\lambda = (\mathbf{a}^\lambda - u)^{-1}g'_\lambda$ belongs to $R[h]$. \square

Next, let R be any k -algebra, and \mathfrak{p} a maximal ideal of R with $\kappa := R/\mathfrak{p}$. For each $f \in R[\mathbf{x}]$, we denote by \bar{f} the image of f in $\kappa[\mathbf{x}]$. The rest of this section is devoted to the proof of the following proposition. This result plays an important role in proving the case (1) of Theorem 2.3 (i).

Proposition 3.2. *Assume that $n = 2$, and let h be a coordinate of $\kappa[\mathbf{x}]$. If h is \mathcal{G} -homogeneous, then there exist $(g_1, g_2) \in C_{\mathcal{G}}(R)$, $a \in \kappa^*$ and $i \in \{1, 2\}$ such that $\det J(g_1, g_2) = 1$ and $h = a\bar{g}_i$.*

When k is an algebraically closed field, Proposition 3.2 easily follows from Lemma 3.1, since $\kappa = k$ is contained in R . In the general case, Proposition 3.2 is proved by using a lifting technique of automorphisms. If $\psi = (f_1, \dots, f_n)$ is an endomorphism of the R -algebra $R[\mathbf{x}]$, then $\bar{\psi} := \text{id}_\kappa \otimes \psi = (\bar{f}_1, \dots, \bar{f}_n)$ is an endomorphism of the κ -algebra $\kappa[\mathbf{x}]$. We remark that $\psi \in \text{Aut}_R R[\mathbf{x}]$ implies $\bar{\psi} \in \text{Aut}_\kappa \kappa[\mathbf{x}]$, since $(\psi_1 \circ \psi_2)^- = \bar{\psi}_1 \circ \bar{\psi}_2$ holds for endomorphisms ψ_1 and ψ_2 of $R[\mathbf{x}]$. We call $\psi \in \text{Aut}_R R[\mathbf{x}]$ a *lift* of $\phi \in \text{Aut}_\kappa \kappa[\mathbf{x}]$ if $\bar{\psi} = \phi$. In general, it is not clear whether every element of $C_{\mathcal{G}}(\kappa)$ has a lift in $C_{\mathcal{G}}(R)$. We say that $\sigma \in \text{Aut}_\kappa \kappa[\mathbf{x}]$ is *elementary* if there exist $1 \leq i \leq n$ and $f \in \kappa[\{x_j \mid j \neq i\}]$ such that

$$\sigma = (x_1, \dots, x_{i-1}, x_i + f, x_{i+1}, \dots, x_n). \quad (3.2)$$

Note that (3.2) is \mathcal{G} -homogeneous if and only if f belongs to $\kappa[\mathbf{x}]_{\gamma_i}$. If this is the case, there exists $g \in R[\mathbf{x}]_{\gamma_i} \cap R[\{x_j \mid j \neq i\}]$ such that $\bar{g} = f$. Then, the elementary automorphism

$$\epsilon = (x_1, \dots, x_{i-1}, x_i + g, x_{i+1}, \dots, x_n) \in C_{\mathcal{G}}(R)$$

is a lift of σ . Clearly, we have $\det J\epsilon = 1$.

Now, let k be any field. For each $\phi = (f_1, f_2) \in \text{Aut}_k k[x_1, x_2]$, we have $\deg \phi := \deg f_1 + \deg f_2 \geq 2$. It is well known (cf. e.g. [3, Thm. 8.5]) that, if $\deg \phi > 2$, then there exist $c \in k^*$ and $l \geq 1$ such that

$$\deg(f_1 - cf_2^l) < \deg f_1 \quad \text{or} \quad \deg(f_2 - cf_1^l) < \deg f_2. \quad (3.3)$$

Using this fact, we can prove the following lemma.

Lemma 3.3. *If $n = 2$, then each $\phi \in C_{\mathcal{G}}(k)$ is written as $\phi = \sigma_1 \circ \dots \circ \sigma_r \circ \tau$ for some $r \geq 0$, where $\sigma_1, \dots, \sigma_r \in C_{\mathcal{G}}(k)$ are elementary, and $\tau \in D_2(k)$.*

Proof. If $\sigma \in C_{\mathcal{G}}(k)$ is elementary and $\tau \in D_2(k)$, then $\tau^{-1} \circ \sigma \circ \tau \in C_{\mathcal{G}}(k)$ is elementary. Hence, it suffices to show that $\phi = \tau \circ \sigma_1 \circ \cdots \circ \sigma_r$ for some τ and $\sigma_1, \dots, \sigma_r$ as in the lemma. We prove this statement by induction on $\deg \phi$.

By assumption, $f_i := \phi(x_i)$ belongs to $k[\mathbf{x}]_{\gamma_i}$ for $i = 1, 2$. First, assume that $\deg \phi = 2$, i.e., $\deg f_1 = \deg f_2 = 1$. If $\gamma_1 \neq \gamma_2$ and $\gamma_1 \neq 0$, then we have $f_1 = ax_1$ and $f_2 = bx_2 + c$ for some $a, b \in k^*$ and $c \in k$, where $c \neq 0$ only if $\gamma_2 = 0$. Since $\phi = (ax_1, bx_2) \circ (x_1, x_2 + c)$ and $(x_1, x_2 + c) \in C_{\mathcal{G}}(k)$, the assertion is true. The case $\gamma_1 \neq \gamma_2$ and $\gamma_2 \neq 0$ is similar. If $\gamma_1 = \gamma_2$, then γ_1 and γ_2 are nonzero, for otherwise $\Gamma_{\mathcal{G}} = \{0\}$, contradicting $\mathcal{G} \neq \{e\}$. Hence, f_1 and f_2 have no constant terms. Thus, ϕ is a linear automorphism. In this case, the assertion follows from linear algebra.

Next, assume that $\deg \phi > 2$. Then, there exist $c \in k^*$ and $l \geq 1$ for which one of the inequalities in (3.3) holds. Since both cases are similar, we assume the former case. In this case, a common monomial appears in f_1 and f_2^l . Since f_i belongs to $k[\mathbf{x}]_{\gamma_i}$ for $i = 1, 2$, it follows that $\gamma_1 = l\gamma_2$. Hence, $\sigma := (x_1 - cx_2^l, x_2)$ belongs to $C_{\mathcal{G}}(k)$. Since ϕ belongs to $C_{\mathcal{G}}(k)$ by assumption, $\phi \circ \sigma = (f_1 - cf_2^l, f_2)$ also belongs to $C_{\mathcal{G}}(k)$. By (3.3), $\deg \phi \circ \sigma$ is less than $\deg f_1 + \deg f_2 = \deg \phi$. Therefore, by induction assumption, we may write $\phi \circ \sigma = \tau \circ \sigma_1 \circ \cdots \circ \sigma_r$, where τ and $\sigma_1, \dots, \sigma_r$ are as in the lemma. Since $\phi = \tau \circ \sigma_1 \circ \cdots \circ \sigma_r \circ \sigma^{-1}$ with $\sigma^{-1} = (x_1 + cx_2^l, x_2) \in C_{\mathcal{G}}(k)$, the assertion holds true. \square

Let us complete the proof of Proposition 3.2. Since κ is an extension field of k , and since h is a \mathcal{G} -homogeneous coordinate of $\kappa[\mathbf{x}]$, there exist $\phi \in C_{\mathcal{G}}(\kappa)$ and $i \in \{1, 2\}$ such that $\phi(x_i) = h$ by Lemma 3.1. By Lemma 3.3, we may write $\phi = \sigma_1 \circ \cdots \circ \sigma_r \circ \tau$ for some $r \geq 0$, where $\sigma_1, \dots, \sigma_r \in C_{\mathcal{G}}(\kappa)$ are elementary, and $\tau = (a_1x_1, a_2x_2)$ with $a_1, a_2 \in \kappa^*$. For $j = 1, \dots, r$, there exists a lift $\epsilon_j \in C_{\mathcal{G}}(R)$ of σ_j with $\det J\epsilon_j = 1$ as mentioned. Then, $(g_1, g_2) := \epsilon_1 \circ \cdots \circ \epsilon_r$ belongs to $C_{\mathcal{G}}(R)$, and satisfies $\det J(g_1, g_2) = 1$. Moreover, we have

$$\phi = \bar{\epsilon}_1 \circ \cdots \circ \bar{\epsilon}_r \circ \tau = (\epsilon_1 \circ \cdots \circ \epsilon_r)^{-} \circ \tau = (\bar{g}_1, \bar{g}_2) \circ \tau = (a_1\bar{g}_1, a_2\bar{g}_2).$$

Therefore, we get $h = \phi(x_i) = a_i\bar{g}_i$. This completes the proof of Proposition 3.2.

4 Case (1) of Theorem 2.3 (i)

The goal of this section is to prove the case (1) of Theorem 2.3 (i). We may assume that $G \neq \{\text{id}\}$. By assumption, there exist $\psi = (f_1, f_2) \in \text{Aut}_K K[\mathbf{x}]$

and a subgroup \mathcal{G} of $(k^*)^2$ with $\mathcal{G} \neq \{e\}$ such that $\psi^{-1} \circ G \circ \psi = \{\delta_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{G}\}$. If ψ belongs to $\text{Aut}_R R[\mathbf{x}]$, then we are done. Note that

$$(\psi \circ \sigma)^{-1} \circ G \circ (\psi \circ \sigma) = \{\delta_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{G}\}$$

holds for each $\sigma \in C_{\mathcal{G}}(K)$. Our strategy is to find $\sigma \in C_{\mathcal{G}}(K)$ such that $\psi \circ \sigma$ belongs to $\text{Aut}_R R[\mathbf{x}]$. There exist $a_1, a_2 \in K^*$ for which $a_1 f_1$ and $a_2 f_2$ belong to $R[\mathbf{x}]$. Since $\sigma_0 := (a_1 x_1, a_2 x_2)$ belongs to $C_{\mathcal{G}}(K)$, by replacing ψ with $\psi \circ \sigma_0 = (a_1 f_1, a_2 f_2)$, we may assume that f_1 and f_2 belong to $R[\mathbf{x}]$. Then, $\det J\psi$ belongs to $R[\mathbf{x}] \cap K^* = R \setminus \{0\}$. Since R is a PID, we may write $\det J\psi = \alpha p_1 \cdots p_m$, where $\alpha \in R^*$ and p_1, \dots, p_m are prime elements of R . Choose ψ so that m is minimal. Then, f_1 and f_2 do not belong to $pR[\mathbf{x}]$ for any prime element p of R . If $m = 0$, then the proof is completed. Indeed, Keller's theorem says that, if $\psi \in \text{Aut}_K K[\mathbf{x}]$ satisfies $\psi(R[\mathbf{x}]) \subset R[\mathbf{x}]$ and $\det J\psi \in R^*$, then ψ restricts to an element of $\text{Aut}_R R[\mathbf{x}]$ (cf. [4, Cor. 1.1.35]). We show that $m = 0$ by contradiction. Suppose that $m \geq 1$, and put $p := p_m$. For $i = 1, 2$, set $g'_i := \psi^{-1}(x_i) \in K[\mathbf{x}]$, and take $b_i \in K^*$ so that $g_i := b_i g'_i$ belongs to $R[\mathbf{x}] \setminus pR[\mathbf{x}]$. Then, $b_i x_i = b_i \psi(g'_i) = \psi(g_i)$ belongs to $R[\mathbf{x}]$, since $\psi(R[\mathbf{x}]) \subset R[\mathbf{x}]$. Hence, b_i is an element of $R \setminus \{0\}$. Since g_1 and g_2 belong to $R[\mathbf{x}]$, we have $b_1 b_2 \det J\psi^{-1} = \det J(g_1, g_2) \in R[\mathbf{x}]$. Hence, $b_1 b_2 = \det J(g_1, g_2) \cdot \det J\psi$ is divisible by $\det J\psi$, and thus by p . Therefore, b_1 or b_2 belongs to pR . In the following, we assume that b_1 belongs to pR .

Since R is a PID over k , we see that $\kappa := R/pR$ is an extension field of k . Consider the endomorphism $\bar{\psi} = (\bar{f}_1, \bar{f}_2)$ of the κ -algebra $\kappa[\mathbf{x}]$. Since f_1 and f_2 do not belong to $pR[\mathbf{x}]$ by assumption, \bar{f}_1 and \bar{f}_2 are nonzero. Since $\mathcal{G} \neq \{e\}$, there exist $\phi \in G$, $i \in \{1, 2\}$ and $\alpha \in k^* \setminus \{1\}$ such that $\phi(f_i) = \alpha f_i$. Then, we have $\bar{\phi}(\bar{f}_i) = (\phi(f_i))^- = (\alpha f_i)^- = \alpha \bar{f}_i \neq \bar{f}_i$. Hence, \bar{f}_i does not belong to κ . This implies that the prime ideal $\ker \bar{\psi}$ of $\kappa[\mathbf{x}]$ is not maximal, and hence is of height at most one. On the other hand, \bar{g}_1 is nonzero by definition, and satisfies $\bar{\psi}(\bar{g}_1) = (\psi(g_1))^- = (b_1 x_1)^- = 0$. Hence, the height of $\ker \bar{\psi}$ is equal to one. Therefore, there exists a prime element q of $\kappa[\mathbf{x}]$ such that $\ker \bar{\psi} = q\kappa[\mathbf{x}]$ (cf. [11, Thm. 20.1]).

Since g_1 is an element of $R[\mathbf{x}]$ which is a coordinate of $K[\mathbf{x}]$, we may find a coordinate h of $\kappa[\mathbf{x}]$ such that \bar{g}_1 belongs to $\kappa[h]$ by the following lemma. This lemma is a consequence of Sathaye [13, Thm. 3].

Lemma 4.1. *Assume that $n = 2$, and let R , p and κ be as above. If $f \in R[\mathbf{x}]$ is a coordinate of $K[\mathbf{x}]$, then there exists a coordinate h of $\kappa[\mathbf{x}]$ such that f belongs to $\kappa[h]$.*

Proof. By assumption, there exists $g \in R[\mathbf{x}]$ such that $K[f, g] = K[\mathbf{x}]$. Since $R' := R_{(p)}$ is a rank-one discrete valuation ring with residue field κ , we obtain

the lemma using the following result of Sathaye [13, Thm. 3] for $A := R'[\mathbf{x}]$ and $(u, v) := (f, g)$: Let R' be a rank-one discrete valuation ring with residue field κ and field of fractions K , and let A be an affine domain over R' such that $A_0 := A \otimes_{R'} K$ and $A_1 := A \otimes_{R'} \kappa$ are polynomial rings in two variables over K and κ , respectively. Take $u, v \in A$ such that $A_0 = K[u, v]$, and let B be the κ -subalgebra of A_1 generated by the images of u and v in A_1 . If B has transcendence degree two over κ , then we have $A_1 = B$. If B has transcendence degree one over κ , then there exist $x, y \in A_1$ such that $A_1 = \kappa[x, y]$ and $B \subset \kappa[x]$. \square

We remark that R' is not assumed to be of “equicharacteristic zero” in the above result of Sathaye, unlike his famous theorem [13, Thm. 1]. When R contains \mathbf{Q} , Lemma 4.1 is also proved by using Rentschler [12] instead of Sathaye [13, Thm. 3] (see the proof at the end of this section).

Now, write $\bar{g}_1 = \Phi(h)$, where $\Phi(x_1) \in \kappa[x_1] \setminus \{0\}$. Then, we have $\Phi(\bar{\psi}(h)) = \bar{\psi}(\bar{g}_1) = 0$. Hence, $\bar{\psi}(h)$ is algebraic over κ . Since $\bar{\psi}(h)$ is an element of $\kappa[\mathbf{x}]$, it follows that $\bar{\psi}(h)$ belongs to κ . Thus, by replacing h with $h - \bar{\psi}(h)$, we may assume that $\bar{\psi}(h) = 0$. Then, h belongs to $\ker \bar{\psi} = q\kappa[\mathbf{x}]$. Since h is a coordinate of $\kappa[\mathbf{x}]$, and hence irreducible, it follows that $h = cq$ for some $c \in \kappa^*$. Therefore, we have $\ker \bar{\psi} = h\kappa[\mathbf{x}]$.

We show that h is \mathcal{G} -homogeneous. Write $h = \sum_{\gamma \in \Gamma_{\mathcal{G}}} h_{\gamma}$, where $h_{\gamma} \in \kappa[\mathbf{x}]_{\gamma}$. Then, for each $\gamma \in \Gamma_{\mathcal{G}}$, we know by Lemma 2.1 that h_{γ} is a k -linear combination of $\delta_{\mathbf{a}}(h)$ for $\mathbf{a} \in \mathcal{G}$. For each $\mathbf{a} \in \mathcal{G}$, we have $\phi_{\mathbf{a}} := \psi \circ \delta_{\mathbf{a}} \circ \psi^{-1} \in G$. Since $\phi_{\mathbf{a}}$ is an element of $\text{Aut}_R R[\mathbf{x}]$ satisfying $\bar{\psi} \circ \bar{\delta}_{\mathbf{a}} = \bar{\phi}_{\mathbf{a}} \circ \bar{\psi}$, we get $\bar{\psi}(\bar{\delta}_{\mathbf{a}}(h)) = \bar{\phi}_{\mathbf{a}}(\bar{\psi}(h)) = \bar{\phi}_{\mathbf{a}}(0) = 0$. Hence, $\delta_{\mathbf{a}}(h)$ belongs to $\ker \bar{\psi}$ for each $\mathbf{a} \in \mathcal{G}$. Thus, h_{γ} belongs to $\ker \bar{\psi} = h\kappa[\mathbf{x}]$ for each $\gamma \in \Gamma_{\mathcal{G}}$. On the other hand, since no common monomials appear in h_{γ} and $h_{\gamma'}$ if $\gamma \neq \gamma'$, we have $\deg h_{\gamma} \leq \deg h$ for each $\gamma \in \Gamma_{\mathcal{G}}$. Hence, $h = h_{\gamma}$ holds for some $\gamma \in \Gamma_{\mathcal{G}}$. Therefore, h is \mathcal{G} -homogeneous. By Proposition 3.2, there exist $\sigma_1 = (h_1, h_2) \in C_{\mathcal{G}}(R)$, $a \in \kappa^*$ and $i \in \{1, 2\}$ such that $\det J\sigma_1 = 1$ and $a\bar{h}_i = h$. Then, $\psi(h_1)$ and $\psi(h_2)$ belong to $R[\mathbf{x}]$. Moreover, $\psi(h_i)$ belongs to $pR[\mathbf{x}]$, since $(\psi(h_i))^- = \bar{\psi}(\bar{h}_i) = \bar{\psi}(a^{-1}h) = 0$. Define $\sigma_2 \in C_{\mathcal{G}}(K)$ by $\sigma_2(x_i) = p^{-1}h_i$ and $\sigma_2(x_j) = h_j$ for $j \neq i$. Then, $(\psi \circ \sigma_2)(x_i) = p^{-1}\psi(h_i)$ and $(\psi \circ \sigma_2)(x_j) = \psi(h_j)$ both belong to $R[\mathbf{x}]$, and

$$\det J(\psi \circ \sigma_2) = \det J\psi \cdot \det J\sigma_2 = p^{-1} \det J\psi \cdot \det J\sigma_1 = \alpha p_1 \cdots p_{m-1}.$$

This contradicts the minimality of m , completing the proof of the case (1) of Theorem 2.3 (i).

Remark: The outline of the proof above is similar to the proof of [8, Thm. 3.2], but more precise treatments, such as Proposition 3.2, are necessary

when k is not algebraically closed. In addition, the proof of [8, Thm. 3.2] uses Sathaye [13, Thm. 1] in a crucial step, which requires that $\text{char } k = 0$.

Finally, we give another proof of Lemma 4.1 in the special case where R contains \mathbf{Q} . Consider the K -derivation $D : K[\mathbf{x}] \ni q \mapsto \det J(f, q) \in K[\mathbf{x}]$. Let $g \in K[\mathbf{x}]$ be such that $K[f, g] = K[\mathbf{x}]$. Then, we have $D(f) = 0$ and $D(g) = \det J(f, g) \in K^*$. Hence, D is *locally nilpotent*, i.e., for each $q \in K[\mathbf{x}]$, there exists $m \geq 1$ such that $D^m(q) = 0$. Take $a \in K^*$ such that $aD(x_i)$ belongs to $R[\mathbf{x}]$ for $i = 1, 2$. Then, $D_1 := aD$ restricts to an R -derivation of $R[\mathbf{x}]$. We may choose a so that $aD(x_1)$ or $aD(x_2)$ does not belong to $pR[\mathbf{x}]$. Then, the κ -derivation $\bar{D} := \text{id}_\kappa \otimes_R D_1$ of $\kappa[\mathbf{x}]$ is nonzero and locally nilpotent. Since $D_1(f) = 0$, we have $\bar{D}(\bar{f}) = 0$. Hence, \bar{f} belongs to $\ker \bar{D}$. On the other hand, there exists a coordinate h of $\kappa[\mathbf{x}]$ such that $\ker \bar{D} = \kappa[h]$ by Rentschler [12] (see also [4, Thm. 1.3.48]), since the field κ contains \mathbf{Q} . This proves Lemma 4.1.

5 Residual variables

Throughout, let R be a k -domain unless otherwise stated, and G a subgroup of $\text{Aut}_R R[\mathbf{x}]$ such that $G_{(0)}$ is diagonalizable. Take $\psi = (f_1, \dots, f_n) \in \text{Aut}_K K[\mathbf{x}]$ and a subgroup \mathcal{G} of $(k^*)^n$ with $\psi^{-1} \circ G \circ \psi = \{\delta_{\mathbf{a}} \mid \mathbf{a} \in \mathcal{G}\}$. We define an algebraic \mathcal{G} -action $V = (V_\gamma)_{\gamma \in \Gamma_{\mathcal{G}}}$ on $R[\mathbf{x}]$ as in (2.2).

Let \mathfrak{p} be a prime ideal of R . We say that G *degenerates* at \mathfrak{p} if there exists $\gamma \in \Gamma_{\mathcal{G}}$ such that $V_\gamma \neq \{0\}$ and $\kappa(\mathfrak{p}) \otimes_R V_\gamma = \{0\}$. If this is the case, G degenerates at any prime ideal of R containing \mathfrak{p} .

The goal of this section is to prove the following theorem.

Theorem 5.1. *Assume that $n \geq 2$, R is a noetherian UFD over k , and G does not degenerate at any maximal ideal of R . Then, Problem 2 has an affirmative answer if at least $n - 1$ of $\Gamma_{\mathcal{G}}^{(1)}, \dots, \Gamma_{\mathcal{G}}^{(n)}$ are not equal to $\Gamma_{\mathcal{G}}$.*

By the following proposition and the remark after Problem 1, Theorem 5.1 implies the case (2) of (i), and (ii) of Theorem 1.1.

Proposition 5.2. *If R is a regular k -domain, then G does not degenerate at any prime ideal of R .*

Proof. Take any prime ideal \mathfrak{p} of R and $\gamma \in \Gamma_{\mathcal{G}}$. We show that $V_\gamma \neq \{0\}$ implies $\kappa(\mathfrak{p}) \otimes_R V_\gamma \neq \{0\}$. Since R is a domain, $R_{\mathfrak{p}} \otimes_R V_\gamma$ contains V_γ . Hence, $V_\gamma \neq \{0\}$ implies $R_{\mathfrak{p}} \otimes_R V_\gamma \neq \{0\}$. Thus, by replacing R and \mathfrak{p} with $R_{\mathfrak{p}}$ and $\mathfrak{p}R_{\mathfrak{p}}$, respectively, we may assume that R is a regular local ring with maximal ideal \mathfrak{p} . We show that $(R/\mathfrak{p}) \otimes_R V_\gamma \neq \{0\}$ by induction on

$r := \dim R$. The assertion is clear if $r = 0$. Assume that $r \geq 1$, and let $a_1, \dots, a_r \in \mathfrak{p}$ be a regular system of parameters of R . Take any $f \in V_\gamma \setminus \{0\}$. Since a regular local ring is a UFD (cf. [11, Thm. 20.3]), there exists $i \geq 0$ such that $f_1 := a_1^{-i}f$ belongs to $R[\mathbf{x}] \setminus a_1R[\mathbf{x}]$. Set $R_1 := R/a_1R$. Then, the image of f_1 in $R_1[\mathbf{x}]$ is nonzero, and belongs to $R_1 \otimes_R V_\gamma$. Note that R_1 is an $(r-1)$ -dimensional regular local ring, and the maximal ideal \mathfrak{p}_1 of R_1 is generated by the images of a_2, \dots, a_r (cf. [11, Thm. 14.2]). Hence, by induction assumption, we get $(R_1/\mathfrak{p}_1) \otimes_R V_\gamma \neq \{0\}$. Since $R_1/\mathfrak{p}_1 \simeq R/\mathfrak{p}$, it follows that $(R/\mathfrak{p}) \otimes_R V_\gamma \neq \{0\}$. \square

The rest of this section is devoted to the proof of Theorem 5.1. First, we investigate the structure of the algebraic \mathcal{G} -action $(K[\mathbf{x}]_\gamma)_{\gamma \in \Gamma_{\mathcal{G}}}$ on $K[\mathbf{x}]$. For $i = 1, \dots, n$, let t_i be the minimal integer $t \geq 1$ with $t\gamma_i \in \Gamma_{\mathcal{G}}^{(i)}$, where $t_i := \infty$ if $\mathbf{Z}\gamma_i \cap \Gamma_{\mathcal{G}}^{(i)} = \{0\}$. Note that $t_i = 1$ if and only if $\Gamma_{\mathcal{G}}^{(i)} = \Gamma_{\mathcal{G}}$. Without loss of generality, we may assume that $t_i = \infty$ if $1 \leq i \leq r$, $2 \leq t_i < \infty$ if $r < i \leq s$ and $t_i = 1$ if $s < i \leq n$ for some $0 \leq r \leq s \leq n$. Set $\Lambda := \sum_{i=r+1}^n \mathbf{Z}\gamma_i$. Then, we have $\Gamma_{\mathcal{G}} = \Lambda \oplus \bigoplus_{i=1}^r \mathbf{Z}\gamma_i$. Moreover, $t_i\gamma_i$ belongs to $\bigcap_{j=r+1}^n \Lambda_j$ for each $r < i \leq n$, where Λ_j is the subgroup of Λ generated by γ_l for $r < l \leq n$ with $l \neq j$. For each $1 \leq i \leq n$, we set $T_i := \mathbf{Z}$ if $t_i = \infty$, and $T_i := \{0, \dots, t_i - 1\}$ if $t_i \neq \infty$. Then, we have the following lemma.

Lemma 5.3. *Each $\gamma \in \Gamma_{\mathcal{G}}$ is uniquely written as*

$$\gamma = \sum_{l=1}^s i_l \gamma_l + \lambda, \quad \text{where } i_l \in T_l \text{ for } l = 1, \dots, s \text{ and } \lambda \in \bigcap_{i=r+1}^n \Lambda_i. \quad (5.1)$$

We have $K[\mathbf{x}]_\gamma = K[\mathbf{x}]_\lambda x_1^{i_1} \cdots x_s^{i_s}$ if $i_1, \dots, i_r \geq 0$, and $K[\mathbf{x}]_\gamma = \{0\}$ otherwise.

Proof. There exist $j_1, \dots, j_n \in \mathbf{Z}$ such that $\sum_{l=1}^n j_l \gamma_l = \gamma$. For each $r < l \leq n$, let q_l and r_l be the quotient and remainder of j_l divided by t_l , respectively. Then, $q_l t_l \gamma_l$ belongs to $\bigcap_{i=r+1}^n \Lambda_i$. Since $r_l = 0$ if $s < l \leq n$, we obtain (5.1) by setting $i_l := j_l$ for $1 \leq l \leq r$, $i_l := r_l$ for $r < l \leq s$, and $\lambda := \sum_{l=r+1}^n q_l t_l \gamma_l$. If $\gamma = \sum_{l=1}^s i'_l \gamma_l + \lambda'$ is another expression, then we have

$$(i_u - i'_u) \gamma_u = \sum_{l \neq u} (i'_l - i_l) \gamma_l + \lambda' - \lambda \in \Gamma_{\mathcal{G}}^{(u)}$$

for each $1 \leq u \leq s$. Since i_u and i'_u belong to T_u , it follows that $i_u = i'_u$ by the definition of t_u . This proves the uniqueness.

Clearly, $K[\mathbf{x}]_\gamma$ contains $K[\mathbf{x}]_\lambda x_1^{i_1} \cdots x_s^{i_s}$ if $i_1, \dots, i_r \geq 0$. Hence, it suffices to check that $K[\mathbf{x}]_\gamma \neq \{0\}$ implies $i_1, \dots, i_r \geq 0$ and $K[\mathbf{x}]_\gamma \subset K[\mathbf{x}]_\lambda x_1^{i_1} \cdots x_s^{i_s}$.

Assume that $x_1^{j_1} \cdots x_n^{j_n}$ belongs to $K[\mathbf{x}]_\gamma$ for some $j_1, \dots, j_n \geq 0$. Then, we have $\sum_{l=1}^n j_l \gamma_l = \gamma$. This implies that $i_l = j_l$ for $1 \leq l \leq r$ by the discussion above. Since $j_l \geq 0$, we get $i_l \geq 0$. Similarly, the quotient q_l of j_l divided by t_l is nonnegative for $r < l \leq n$. Hence, $m := \prod_{l=r+1}^n x_l^{q_l t_l}$ belongs to $K[\mathbf{x}]_\lambda$. Therefore, $x_1^{j_1} \cdots x_n^{j_n} = m x_1^{i_1} \cdots x_s^{i_s}$ belongs to $K[\mathbf{x}]_\lambda x_1^{i_1} \cdots x_s^{i_s}$. \square

We say that $f \in R[\mathbf{x}] \setminus \{0\}$ is *primitive* if no prime element p of R satisfies $f \in pR[\mathbf{x}]$. We remark that, if R is a UFD and $f \in R[\mathbf{x}] \setminus \{0\}$ is primitive, then $Bf \cap R[\mathbf{x}] = (B \cap R[\mathbf{x}])f$ holds for any K -submodule B of $K[\mathbf{x}]$. In the situation of Theorem 5.1, we may assume that f_1, \dots, f_n are primitive elements of $R[\mathbf{x}]$. Write $\gamma \in \Gamma_{\mathcal{G}}$ as in (5.1), and assume that $i_1, \dots, i_r \geq 0$. Then, since $f_1^{i_1} \cdots f_s^{i_s}$ is primitive, we know by Lemma 5.3 that

$$\begin{aligned} V_\gamma &= \psi(K[\mathbf{x}]_\gamma) \cap R[\mathbf{x}] = \psi(K[\mathbf{x}]_\lambda x_1^{i_1} \cdots x_s^{i_s}) \cap R[\mathbf{x}] \\ &= \psi(K[\mathbf{x}]_\lambda) f_1^{i_1} \cdots f_s^{i_s} \cap R[\mathbf{x}] = (\psi(K[\mathbf{x}]_\lambda) \cap R[\mathbf{x}]) f_1^{i_1} \cdots f_s^{i_s} \quad (5.2) \\ &= V_\lambda f_1^{i_1} \cdots f_s^{i_s}. \end{aligned}$$

Now, take any R -algebra S , and let $\bar{f}_1, \dots, \bar{f}_n$ be the images of f_1, \dots, f_n in $S[\mathbf{x}]$. We consider the algebraic \mathcal{G} -action $V_S = (S \otimes_R V_\gamma)_{\gamma \in \Gamma_{\mathcal{G}}}$ on $S[\mathbf{x}]$. From (5.2), it follows that $S \otimes_R V_\gamma = (S \otimes_R V_\lambda) \bar{f}_1^{i_1} \cdots \bar{f}_s^{i_s}$ for each $\gamma \in \Gamma_{\mathcal{G}}$ with $i_1, \dots, i_r \geq 0$.

In the notation above, we have the following proposition.

Proposition 5.4. *Assume that V_S is diagonalizable, and $S \otimes_R V_\gamma \neq \{0\}$ holds for each $\gamma \in \Gamma_{\mathcal{G}}$ with $V_\gamma \neq \{0\}$. Then, $\bar{f}_1, \dots, \bar{f}_s$ form a partial system of coordinates of $S[\mathbf{x}]$.*

Proof. By assumption, there exists $\sigma \in \text{Aut}_S S[\mathbf{x}]$ such that $y_i := \sigma(x_i)$ is V_S -homogeneous for $i = 1, \dots, n$. We show that, for each $1 \leq l \leq s$, there exist $1 \leq \sigma(l) \leq n$ and $\alpha_l \in S[\mathbf{x}]^*$ satisfying $\bar{f}_l = \alpha_l y_{\sigma(l)}$. First, we claim that there exists $1 \leq \sigma(l) \leq n$ for which $y_{\sigma(l)}$ is written as $\bar{f}_1^{i_1} \cdots \bar{f}_s^{i_s} g$ with $i_l \geq 1$, where $i_u \in T_u$ for each $1 \leq u \leq s$, and $g \in S \otimes_R V_\lambda$ for some $\lambda \in \bigcap_{i=r+1}^n \Lambda_i$. In fact, if not, $S[\mathbf{x}] = S[y_1, \dots, y_n]$ is contained in $\bigoplus_{\gamma \in \Gamma_{\mathcal{G}}^{(l)}} S \otimes_R V_\gamma$. Since γ_l does not belong to $\Gamma_{\mathcal{G}}^{(l)}$ if $1 \leq l \leq s$, we have $S \otimes_R V_{\gamma_l} = \{0\}$. This implies that $V_{\gamma_l} = \{0\}$ by assumption, contradicting $f_{\gamma_l} \in V_{\gamma_l}$. It remains only to check that $i_l = 1$, $i_t = 0$ for each $t \neq l$, and g is a unit of $S[\mathbf{x}]$. Take any prime ideal \mathfrak{p} of S , and let $\pi : S[\mathbf{x}] \rightarrow (S/\mathfrak{p})[\mathbf{x}]$ be the natural surjection. Then, $\pi(y_{\sigma(l)}) = \pi(\bar{f}_1)^{i_1} \cdots \pi(\bar{f}_s)^{i_s} \pi(g)$ is a coordinate of $(S/\mathfrak{p})[\mathbf{x}]$, and hence is an irreducible element of $(S/\mathfrak{p})[\mathbf{x}]$. Now, consider the algebraic \mathcal{G} -action $V_{S/\mathfrak{p}}$ on $(S/\mathfrak{p})[\mathbf{x}]$. Then, $\pi(\bar{f}_i)$ belongs to $(S/\mathfrak{p}) \otimes_R V_{\gamma_i}$ for each i . If $1 \leq i \leq s$, then $((S/\mathfrak{p}) \otimes_R V_{\gamma_i}) \cap (S/\mathfrak{p})$ equals $\{0\}$, since $\gamma_i \neq 0$. Hence, we have either

$\pi(\bar{f}_i) = 0$ or $\pi(\bar{f}_i) \notin S/\mathfrak{p}$. By the irreducibility of $\pi(y_{\sigma(l)})$, it follows that $i_l = 1$, $i_t = 0$ for each $t \neq l$, and $\pi(g)$ belongs to $(S/\mathfrak{p})[\mathbf{x}]^* = (S/\mathfrak{p})^*$. Since \mathfrak{p} is any prime ideal of S , we know that the constant term c of g is a unit of S , and $g - c$ is a nilpotent element of $S[\mathbf{x}]$. Therefore, g is a unit of $S[\mathbf{x}]$. \square

Let us complete the proof of Theorem 5.1. We may assume that $\Gamma_{\mathcal{G}}^{(i)} \neq \Gamma_{\mathcal{G}}$ for each $i \neq n$. Then, we have $t_i \neq 1$ for each $i \neq n$. Thanks to Lemma 2.4, it suffices to show that f_1, \dots, f_{n-1} form a partial system of coordinates of $R[\mathbf{x}]$. Take any prime ideal \mathfrak{p} of R . By assumption, G does not degenerate at maximal ideals of R containing \mathfrak{p} . Hence, G does not degenerate at \mathfrak{p} . Moreover, $G_{\mathfrak{p}}$ is diagonalizable by the assumption of Problem 2. Hence, by Proposition 5.4, the images of f_1, \dots, f_{n-1} in $\kappa(\mathfrak{p})[\mathbf{x}]$ form a partial system of coordinates of $\kappa(\mathfrak{p})[\mathbf{x}]$. This implies that f_1, \dots, f_{n-1} form a partial system of coordinates of $R[\mathbf{x}]$ thanks to the result on “residual variables” by Bhatwadekar-Dutta [2, Remark 3.4], since R is a noetherian UFD by assumption, and UFD is seminormal.

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